Given $X \xrightarrow{f} Y$ we want to be able to define the derivatives:

$$df : T_X \to f^*T_Y. \quad \text{(tangent bundle)}$$

resp: $df : f^*\Omega_Y \to \Omega_X$. (cotangent bundles).

Kähler Differentials

Let $A \to B$ be a ring hom.
M a B-module. An A-derivation of B into M is a

$$d : B \to M$$

such that
$d(bb') = b\,db' + b'\,db$. (Leibniz).
$d(b+b') = db + db'$ (additive)
$d\,a = 0$ (trivial on A).

NOTE: d is A-linear.

$(\ d(ab) = a\,db + b\,da = a\,db).$

Defn The B-module of relative Kähler differentials:

$$\Omega_{B/A} := \bigoplus_{b \in B} B \cdot da \Big/$$

- additivity
- Leibniz
- triviality on A.

$\exists$ a canonical A-derivation:

$$d : B \longrightarrow \Omega_{B/A}.$$

(and any A-derivation factors through b)

$$b_1 \otimes b_2 \longrightarrow b_1 \cdot b_2$$

Defn #2. Let $B \underset{A}{\otimes} B \xrightarrow{\Delta} B$ be the diagonal homomorphism.

- Set $I = \ker \Delta$.

  I/I^2 is a B-module.

  $d : B \to I/I^2$

  $b \longrightarrow 1 \otimes b - b \otimes 1$.

CLAIM: $d : B \to I/I^2$ is an A-derivation and there is an isom.

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "d"'] & \Omega_{B/A} \arrow[d, "\wr"] \\ & I/I^2 \end{tikzcd}$$

- Leibniz: $b_1 b_2 \longrightarrow \underline{b_1 b_2} \otimes 1 - 1 \otimes \underline{b_1 b_2}$.

  $- b_1 \cdot (b_2 \otimes 1 - 1 \otimes b_2) = \underline{b_1 b_2} \otimes 1 - \underline{b_1 \otimes b_2}$

  $- b_2 (b_1 \otimes 1 - 1 \otimes b_1) = b_2 b_1 \otimes 1 - b_2 \otimes b_1$.

  $(b_1 \otimes b_2 - 1 \otimes b_1 b_2 - b_2 b_1 \otimes 1 + b_2 \otimes b_1)$

  $\shortparallel$

  $- (b_1 \otimes 1 - 1 \otimes b_1)(b_2 \otimes 1 - 1 \otimes b_2) \in I^2$.

- $\Omega_{B/A} \to I/I^2$ surjective.

  Enough to show $\langle b \otimes 1 - 1 \otimes b \rangle_{b \in B} = I$.

  Suppose: $\sum x_i \otimes y_i \in \ker$

  $\Rightarrow \sum x_i y_i = 0$

$$\sum x_i \otimes y_i = \sum (x_i \otimes y_i - x_i y_i \otimes 1)$$
$$= \sum x_i \otimes 1 (1 \otimes y_i - y_i \otimes 1) \in \langle b \otimes 1 - 1 \otimes b \rangle.$$

- Want to construct:

$$I/I^2 \to \Omega_{B/A}.$$

First $\exists$ a map:

$$B \otimes_A B \longrightarrow \Omega_{B/A}.$$
$$b_1 \otimes b_2 \longrightarrow b_1 d b_2$$

This defines:

$$I \longrightarrow \Omega_{B/A}.$$
$$1 \otimes b - b \otimes 1 \longrightarrow db.$$

Moreover:

$$(1 \otimes b_1 - b_1 \otimes 1) \cdot (1 \otimes b_2 - b_2 \otimes 1) \longrightarrow db_1 b_2 - b_1 db_2 - b_2 db_1 = 0.$$

$$\Rightarrow I^2 \longrightarrow 0.$$

Thus: $$\begin{tikzcd}\Omega_{B/A} \arrow[r, two heads] \arrow[rr, bend right, "= \mathrm{Id}"'] & I/I^2 \arrow[r] & \Omega_{B/A}\end{tikzcd} \Rightarrow \Omega_{B/A} \cong I/I^2.$$

$\blacksquare$

Proposition.

Fiber Product A. Let
$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & D = B \underset{A}{\otimes} C . \end{tikzcd}$$
Then $\Omega_{D/C} = \Omega_{B/A} \underset{B}{\otimes} D$.

Localization B. If S is a mult. system in B then:
$$\Omega_{S^{-1}B/A} = S^{-1}\Omega_{B/A}.$$

C. If $B = A[x_1, \dots, x_n]$, then
$$\Omega_{B/A} = B\,dx_1 \oplus \cdots \oplus B\,dx_n .$$

D. Let $A \to B \to C$ be rings & homs.
There is a natural exact sequence:
$$\Omega_{B/A} \underset{B}{\otimes} C \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0 .$$
"relative cotangent sequence".

E. Let $B \to \underset{\overset{\shortparallel}{C}}{B/I}$. There is a natural exact sequence of C-modules:
$$I/I^2 \longrightarrow \Omega_{B/A} \underset{B}{\otimes} C \longrightarrow \Omega_{C/A} \longrightarrow 0$$
"conormal sequence".
$$f \longrightarrow d\bar{f} \otimes 1.$$
$$\left(f \to d(\bar{f} + g \cdot h) \otimes 1 = d\bar{f} + \underbrace{dg \otimes h + dh \otimes g}_{=0.} \right).$$

Cor. If B is a localization of a f.g.'d A-algebra then $\Omega_{B/A}$ is finitely generated.

Proof of (C) → (E).

(C) $d(x_1^{l_1} \cdot x_2^{l_2} \cdots x_n^{l_n}) = l_1 x_1^{l_1-1} x_2^{l_2} \cdots x_n^{l_n} dx_1 + \dots + l_n x_1^{l_1} \cdots x_n^{(l_n-1)} dx_n$

the map: $B \xrightarrow{d} Bdx_1 + \dots + Bdx_n$

is a derivation. Can show:

$$B \to Bdx_1 \oplus \dots \oplus Bdx_n$$

satisfies the universal property $\Rightarrow \cong \Omega_{B/A}$.

(D) $A \to B \to C$

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

1. all C-mods.
2. $d: C \to \Omega_{C/B}$ is an A-derivation

$\Rightarrow \exists$ a map:

$$\Omega_{C/A} \to \Omega_{C/B}$$

the generators of $\Omega_{C/B}$ are $d(c)$.

gens. of $\Omega_{C/A}$ are $d(c)$.

$\Rightarrow$ surj.

3. $$\begin{tikzcd} \Omega_{B/A} \otimes_B C \arrow[r] \arrow[d, equal] & \Omega_{C/A} \arrow[rr] \arrow[d, equal] & & \Omega_{C/B} \arrow[d, equal] \\ \bigoplus_{b \in B} C\,db \Big/ \substack{\text{Leibniz} \\ + \text{ADD} \\ + \text{constants} \\ \text{in } B} \arrow[r] & \bigoplus_{c \in C} C\,dc \Big/ \substack{\text{Leibniz} \\ + \text{ADD.} \\ + \text{constants.} \\ \text{in } A.} \arrow[rr] & & \bigoplus_{c \in C} C\,dc \Big/ \substack{\text{Leibniz} \\ + \text{add.} \\ + \text{constants} \\ \text{in } B} \end{tikzcd}$$

ONLY difference: constants in A v. B.

(E) $B \to B/I = C$.

$$\begin{tikzcd} C\otimes_B \Omega_{B/A} \arrow[r] \arrow[d, equal] & \Omega_{C/A} \arrow[r] \arrow[d, equal] & 0. \\ \bigoplus_{b\in B} C\,db \Big/ \text{Leibniz + add + A-const.} \arrow[r] & \bigoplus_{c\in C} C\cdot dc \Big/ \text{Leibniz add A-constants.} & \end{tikzcd}$$

ker gen'd by image of

$$\begin{tikzcd} & \bigoplus_{b\in I} C\,db \arrow[r] & C\otimes_B \Omega_{B/A}. \\ & & df. \\ I/I^2 \arrow[uurr] & & \\ f \arrow[uurr, mapsto] & & \end{tikzcd}$$

Also have:

Let $\sum_{b_i\in I} c_i\,db_i \in \ker$.

Let $\bar{c}_i \in B$ w/ image c_i

$$\sum \bar{c}_i b_i \in I \longmapsto \sum_{b_i\in I} c_i\,db_i. \qquad \blacksquare$$

EXAMPLE. $\mathbb{A}^n_k \supset (f=0) = X$.

$\Omega_{\mathbb{A}^n/k} = \bigoplus_{i=1}^n k[x_1,\dots,x_n]\,dx_i$ free of rank $= n$.

$$(f)/(f^2) \longrightarrow \bigoplus_{i=1}^n k[x_1,\dots,x_n]/(f)\,dx_i \longrightarrow \Omega_{X/k} \longrightarrow 0$$

Theorem:

A. Let B be a local k-algebra w/ $B/m \cong k$.

Then $m/m^2 \cong \Omega_{B/k} \otimes_B k$.

B. Assume k is perfect and B is a local k-algebra w/ $B/m \cong k$.

$\Omega_{B/k}$ is free w/ rank = dim B $\iff$ B is a regular local ring.

Proof. Ⓐ. We have maps:

$$\begin{tikzcd}
m/m^2 \arrow[r] & k \otimes \Omega_{B/k} \arrow[r] & \Omega_{k/k} \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & 0. \\
 & & 0 &
\end{tikzcd}$$

Shows $m/m^2 \twoheadrightarrow k \otimes \Omega_{B/k}$.

The other way:

• WANT to SHOW •

$$\begin{tikzcd}
\mathrm{Hom}_k(k \otimes \Omega_{B/k}, k) \arrow[r, two heads] \arrow[d, phantom, "\cong" rotate=-90] & \mathrm{Hom}(m/m^2, k) \\
\mathrm{Hom}_B(\Omega_{B/k}, k) = \begin{pmatrix} \text{derivations} \\ \partial : B \to k \end{pmatrix}. &
\end{tikzcd}$$

To show surj.:

Let $\varphi \in \mathrm{Hom}(m/m^2, k)$.

Define a derivation:

$$\partial : B \to k.$$

$$b = \lambda + f_1 + f_2 \to \varphi(f_1).$$

$\lambda \in k$

$f_1 \in m$ (N.T.S. ind. of choice).

$f_2 \in m^2$.

Ⓑ. (⇒): $\Omega_{B/k}$ free w/ rk = dim B

⇒ m/m^2 has finite k-dim'n = dim B.

⇒ B is a regular local ring.

(⇐): B a reg. local ring.

⇒ $\dim \Omega_{B/k} \otimes k = \dim B$.

If $K = \text{Frac}\, B$, then:

$$\Omega_{K/k} = \Omega_{B/k} \otimes_B K.$$

Need 2 Lemmas

Lemma A. If K/k is a f.g.'d separable extn. then

$$\dim_K(\Omega_{K/k}) = \text{tr.deg}(K/k)$$

Lemma B. If M is a f.g.'d module over B w/ $\dim_K(M \otimes_B K) = \dim_k(M \otimes_B k)$. Then M is free of rk = r.

■

EXAMPLE $\mathbb{F}_p(x)[y]/(y^p - x) = B$.

↖ irreducible polynomial. ⇒ maximal.

$$\begin{tikzcd}
(y^p - x) \arrow[r] & B\,dy \arrow[r] \arrow[d, equal] & \Omega_{B/k(x)} \arrow[r] & 0 \\
& B \otimes \Omega_{\mathbb{A}^1/k(x)} & &
\end{tikzcd}$$

Definition Let $f: X \to Y$ be a map of schemes. Let $\Delta: X \to X \times_Y X$ be the diagonal map with ideal I. Define the relative cotangent sheaf to be:

$$\Omega_{X/Y} := \Delta^*(I/I^2).$$

All the previous results generalize to schemes.

THEOREM

BASE CHANGE:

$$\begin{tikzcd} X \times_Y Z \arrow[r, "f'"] \arrow[d, "g'"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$$\Omega_{X \times_Y Z / Z} \cong g'^*(\Omega_{X/Y}).$$

Relative Cotangent SEQUENCE: $X \xrightarrow{f} Y \xrightarrow{g} Z$. $\exists$ an exact sequence:

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

Relative CONORMAL SEQUENCE: $X \xrightarrow{f} Y$ map of schemes. $Z \overset{i}{\subset} X$ a closed subscheme w/ ideal I. $\exists$ an exact sequence:

$$I/I^2 \to i^*\Omega_{X/Y} \to \Omega_{Z/Y} \to 0.$$

Affine SPACE: $\Omega_{\mathbb{A}^n/Y} \cong \mathcal{O}_{\mathbb{A}^n_Y} dx_1 \oplus \cdots \oplus \mathcal{O}_{\mathbb{A}^n_Y} dx_n.$

Affine subsets If

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \mathcal{U} = \operatorname{Spec} B \arrow[u, phantom, "\cup"] \arrow[r] & \operatorname{Spec} A \arrow[u, phantom, "\cup"] \end{tikzcd}$$

then $\Omega_{X/Y}|_{\mathcal{U}} = \widetilde{\Omega}_{B/A}.$

If X/k a variety of dim $= n$ say X is **smooth** if $\Omega_{X/k}$ is locally free of rank n.

If so call $\Omega_{X/k}$ the **cotangent bundle** & define the tangent bundle to be:

$$T_X := \operatorname{Hom}(\Omega_{X/k}, \mathcal{O}).$$

NOTE: Given a map of smooth varieties

$$\pi : X \to Y$$

we have:

$$\pi^* \Omega_Y \to \Omega_X$$

AND (dually) $T_X \to \pi^* T_Y.$

If $Y \overset{i}{\hookrightarrow} X$ is a closed immersion of smooth varieties then

$$0 \to \underbrace{I/I^2}_{\text{the conormal bundle.}} \to \Omega_X|_Y \to \Omega_Y \to 0$$

The dual sequence:

$$0 \to T_Y \to T_X|_Y \to \overset{\mathcal{H}om_{\mathcal{O}_Y}(I/I^2, \mathcal{O}_Y)}{\overset{\|}{N_{X/Y}}} \to 0$$

defines the normal bundle.

If Y is a divisor in X then: $I_Y \cong \mathcal{O}_X(-Y)$ (invertible!)

So: $$I_Y/I_Y^2 \cong \mathcal{O}_X(-Y)|_Y =: \mathcal{O}_Y(-Y).$$

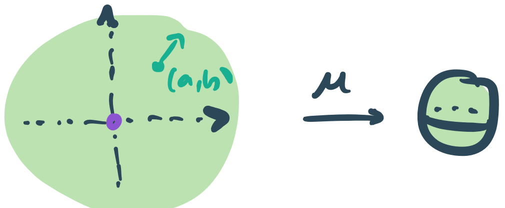
AND: $$N_{Y/X} \cong \mathcal{O}_Y(Y).$$

Theorem. (the Euler sequence) $Y = \operatorname{Spec} A$.

There is an exact sequence of sheaves on X:

$$\begin{tikzcd}
0 \arrow[r] & \Omega_{\mathbb{P}^n/Y} \arrow[r] & \mathcal{O}_X(-1)^{\oplus n+1} \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \mathcal{O}_X \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & 0. \\
 & & e_i \arrow[r] & x_i &
\end{tikzcd}$$

EXAMPLE. $n = 1$. $Y = \operatorname{Spec} \mathbb{C}$.



$$\mathbb{A}^2 \setminus \{0\} \longrightarrow \mathbb{CP}^1$$

$$\begin{tikzcd}
 & 1 \arrow[r] & x\frac{\partial}{\partial x} \oplus y\frac{\partial}{\partial y}. & & \\
0 \arrow[r] & \mathcal{O} \arrow[r] & T_{\mathbb{A}^2\setminus\{0\}} \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] & \mu^* T_{\mathbb{CP}^1} \arrow[r] & 0 \\
 & & T_{\mathbb{A}^2}\big|_{\mathbb{A}^2\setminus\{0\}} \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] & & \\
 & & \mathcal{O}_{\mathbb{A}^2}\frac{\partial}{\partial x} \oplus \mathcal{O}_{\mathbb{A}^2}\frac{\partial}{\partial y} & &
\end{tikzcd}$$

Only issue: this sequence is defined on $\mathbb{A}^2 \setminus 0$ not $\mathbb{CP}^1$.

Proof of Euler Sequence

$$\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0.$$

(twist of $\mathcal{O}^{\oplus n+1} \longrightarrow \mathcal{O}(1)$ by $\mathcal{O}(-1)$

$e_i \longrightarrow x_i$

$\uparrow$

these sections generate!

$\Rightarrow$ surjective)

Let K be the kernel. W.T.S. $K \cong \Omega_{\mathbb{P}^n_A/y}$.

A. Looking at $1/x_i$-chart, see:

$$\begin{matrix} A[x_1/x_i, \dots, x_n/x_i]\cdot 1/x_i \\ \oplus \\ \vdots \\ \oplus \\ A[x_1/x_i, \dots, x_n/x_i]\cdot 1/x_i \end{matrix} \longrightarrow A[x_1/x_i, \dots, x_n/x_i].$$

$$e_j \longrightarrow x_j/x_i.$$

$$K(\mathbb{A}^n_i) = \left\langle \frac{1}{x_i}e_j - \left(x_j/x_i^2\right)e_i \;\left(=: d\left(x_j/x_i\right)\right) \right\rangle$$

(free of rank n) $\cong \Omega_{\mathbb{A}^n_i/A}$.

B. Transitions.

For each $\mathbb{A}^n_i$ have identified

$$\Omega_{\mathbb{P}^n/A}(\mathbb{A}^n_i) \underset{\psi_i}{\cong} K(\mathbb{A}^n_i)$$ let $\mathbb{A}^n_{ij} = \mathbb{A}^n_i \cap \mathbb{A}^n_j$.

Need to show:

$$\begin{tikzcd}
d\left(\frac{x_j}{x_i}\right) \arrow[rrr, squiggly, "\psi_i"] \arrow[ddd, "\varphi_J"'] & & & \frac{1}{x_i}e_j - \frac{x_j}{x_i^2}e_i \arrow[ddd, squiggly] \\
& \Omega_{\mathbb{P}^n/A}(\mathbb{A}^n_{il}) \arrow[r, phantom, "\underset{\psi_i|_{\mathbb{A}^n_{il}}}{\cong}"] \arrow[d, "\cong"', "\varphi_J"] \arrow[dr, phantom, "\circlearrowright"] & K|_{\mathbb{A}^n_{il}} \arrow[d, "\cong"', "\varphi_K"] & \\
& \Omega_{\mathbb{P}^n/A}(\mathbb{A}^n_{il}) \arrow[r, phantom, "\underset{\psi_l|_{\mathbb{A}^n_{il}}}{\cong}"] & K|_{\mathbb{A}^n_{il}} & \\
\frac{x_l}{x_i} d\left(\frac{x_j}{x_l}\right) + \frac{x_j}{x_l} d\left(\frac{x_l}{x_i}\right) & & & \frac{x_l}{x_i}\frac{1}{x_l}e_j - \frac{x_j x_l}{x_i^2}\cdot\frac{1}{x_l}e_l
\end{tikzcd}$$

$$=$$

$$\frac{x_l}{x_i} d\left(\frac{x_j}{x_l}\right) - \frac{x_j}{x_l}\cdot\left(\frac{x_l}{x_i}\right)^2 d\left(\frac{x_i}{x_l}\right)$$

$$\psi_l \rightsquigarrow \quad \frac{x_l}{x_i}\frac{1}{x_l}e_j - \frac{x_l}{x_i}\cdot\frac{x_j}{x_l^2}\overset{\times}{e_l} \qquad \checkmark$$

$$- \frac{x_j}{x_l}\left(\frac{x_l}{x_i}\right)^2\left(\frac{1}{x_l}e_i - \frac{x_i}{x_l^2}\overset{\times}{e_l}\right) \qquad \blacksquare$$

Let 1 $k = \bar{k}$

2 X/k a smooth k-variety

3 $Y \subset X$ a closed subscheme.

Then Y is smooth if

A $\Omega_{Y/k}$ is locally free w/ $\mathrm{rk} = \dim Y$

OR B $\Omega_{Y/k}$ is locally free and

$$0 \to I/I^2 \to \Omega_{X/k}|_Y \to \Omega_{Y/k} \to 0.$$

is exact.

(In this case: I/I^2 is locally free w/ $\mathrm{rk} = r := \operatorname{codim} Y \subset X$ AND I is locally generated by r elements)

COROLLARIES. $/k = \bar{k}$

I If Y is a variety, $\exists\, \emptyset \neq U \subseteq Y$ such that U is nonsingular.

II If $Y \subset X$ is a reduced divisor X is smooth then:

Y smooth $\iff$ $\Omega_{Y/k}$ is locally free.

III If $Y \subset \mathbb{A}^N_k$ is an irred. affine scheme of codimension m with defining ideal:

$$I = (f_1, \dots, f_\ell)$$

then Y is smooth $\Longleftrightarrow$ the Jacobian matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & & \\ \frac{\partial f_1}{\partial x_2} & \ddots & \\ \vdots & & \\ \frac{\partial f_1}{\partial x_N} & & \end{bmatrix}$$

has rank $= m$ at every point $y \in Y$.

Sketch of Corollaries

I. k-perfect $\Rightarrow \Omega_{K(Y)/k}$ has rank $= \dim Y$

But $\Omega_{K(Y)/k}$ is the localization of Ω_Y at the generic pt

$\Rightarrow \exists$ an open set $\mathcal{U}'$ s.t.
rk $\Omega_{\mathcal{U}'/k} = \dim Y$.

$\Rightarrow \exists$ an open set $\emptyset \neq \mathcal{U} \subset \mathcal{U}'$ s.t.
$\Omega_{\mathcal{U}/k}$ is locally-free. ✓

II. If Y is a divisor

$\Rightarrow I = \mathcal{O}(-Y)$

$\Rightarrow I/I^2$ is invertible.

$(\cong \mathcal{O}_Y(-Y))$

$$\Rightarrow \underset{\underset{\text{invertible}}{\uparrow}}{I/I^2} \xrightarrow{\partial} \underset{\underset{\text{vector bundle.}}{\uparrow}}{\Omega_X|_Y}$$

$m \in I/I^2(U) \setminus 0 \Rightarrow m$ non-zero on some component (b.c. Y is reduced)

$\Rightarrow m \neq 0$ at the generic pt.

$\Rightarrow \partial(m) \neq 0$ (b.c. $\exists$ an open set where the component is smooth) ✓

III. The Jacobian mx:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \\ \vdots & & \\ & & \ddots \\ \frac{\partial f_1}{\partial x_N} & & \end{bmatrix} = J(x_1, \ldots, x_n)$$

is the map:

$$I/I^2 \longrightarrow \mathcal{O}_Y dx_1 \oplus \cdots \oplus \mathcal{O}_Y dx_n.$$

At a pt $\bar{y} \in Y$

$$n - \operatorname{rk} J(\bar{y}) = \dim_k(\Omega_Y|_{\bar{y}}).$$ ✓

Let $X \subset \mathbb{P}^n_k$ be a proj. variety with $k = \bar{k}$.

Defn: The projective tangent plane to X at a point $p \in X$ is the intersection of all planes tangent to X at p, denoted:

non-standard $\rightsquigarrow$ $\mathbb{T}_p X \subset \mathbb{P}^n$.

Lemma: Suppose $X = (f_1, \dots, f_\ell)$.

Then
$$\mathbb{T}_p(X) = \left(\frac{\partial f_i}{\partial x_0}(p) x_0 + \dots + \frac{\partial f_i}{\partial x_n}(p) x_n = 0 \right) \subseteq \mathbb{P}^n_k.$$
$$(1 \leq i \leq \ell)$$

Proof: Assume for simplicity $p = [1:0:\dots:0]$.

Work on the chart: $\mathbb{A}^n_0$ $(1/x_0)$.

$\Rightarrow$ $f_i(1, y_1, \dots, y_n)$ has no constant term.

Locally the tangent space is defined by:

$$\frac{\partial f_i}{\partial y_1}(0,\dots,0) y_1 + \dots + \frac{\partial f_i}{\partial y_n}(0,\dots,0) y_n = 0.$$

Can check this is the dehomog. of

$$\left(\frac{\partial f_i}{\partial x_1}(1,0,\dots,0) x_1 + \dots + \frac{\partial f_i}{\partial x_n}(1,0,\dots,0) x_n = 0 \right).$$ ✓

Cor. (Projective Jacobian criterion)

$$\operatorname{rk}\begin{bmatrix} \frac{\partial f_1}{\partial x_0}(p) & \frac{\partial f_1}{\partial x_2}(p) & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_0}(p) & \cdots & & \\ \vdots & & & \end{bmatrix} = n - \dim(m/m^2).$$

Bertini's Theorem

Let $X \subset \mathbb{P}^n_k$ be a nonsingular closed subvariety$/k = \bar{k}$.

$\exists$ a hyperplane $H \subsetneq \mathbb{P}^n_k$ not containing X such that the scheme $H \cap X$ is regular at every point. The set of such hyperplanes forms an open set of the complete linear system: $|H \cap X|$.

Proof. Let $(\mathbb{P}^n_k)^\vee = |H|$ be the dual projective space.

We define a closed subvariety
$$B \subset X \times (\mathbb{P}^n_k)^\vee.$$

Set theoretically:

$$B = \{(x,H) \mid H \supset X \text{ or } H \cap X \text{ is singular at } x\}.$$

GOAL: Show the image of B in $(\mathbb{P}^n_k)^\vee$ has dimn $\leq n-1$.

↑ coords $[a_0 : \cdots : a_n]$).

Step 1. Suppose:

$$X = (f_1 = \cdots = f_d = 0)$$

f_i homogeneous.

Define

$$B \subset X \times (\mathbb{P}^{n\vee}_k) = (f_1 = \cdots = f_d = 0) \subseteq \mathbb{P}^n_k \times (\mathbb{P}^n_k)^\vee.$$

$$B = (f_1(x) = \cdots = f_d(x) = 0, \underbrace{a_0x_0 + \cdots + a_nx_n = 0}_{\text{"}p \in H\text{"}},$$

AND:

$$n+1 \updownarrow \qquad \operatorname{rank} \begin{bmatrix} \frac{\partial f_1}{\partial x_0} & \cdots & \frac{\partial f_d}{\partial x_0} & a_0 \\ \vdots & & \vdots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_d}{\partial x_n} & a_n \end{bmatrix} \leq n - \dim X.$$

(eqns! ↗)

CLAIM: these cut out B set-theoretically.

$*$ X smooth $\Longrightarrow$ m/m^2 has dim n d

$$\Longrightarrow \left\langle \frac{\partial f_i}{\partial x_0}(p)x_0 + \dots + \frac{\partial f_i}{\partial x_n}(p)x_n \right\rangle$$

have rank $n - \dim X$ at p

$**$ $\operatorname{rk}\left(\operatorname{Jact}\begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}\right) \leq n - \dim X \Rightarrow a_0x_0 + \dots + a_nx_n$ in the span of above $\Rightarrow H \cap X$ has dim X OR $H \cap X$ singular at $p \in X$.

Step 2 Showing $\dim B \leq n-1$.

Fix a pt $x \in X$. What is the fiber of B over x?

$$B_x = \left\{ \begin{array}{c} \text{Hyperplanes tangent to} \\ x \in X \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \text{Hyperplanes containing} \\ \text{projectivized tangent} \\ \text{plane} \end{array} \right\}$$

$$\cong \mathbb{P}^{n - \dim X - 1}$$

$\Rightarrow B$ has dimension $n - \dim X - 1 + \dim X$
$= n - 1.$

$\Rightarrow \exists$ a hyperplane $H \in (\mathbb{P}^n_k)^\vee$ s.t. $H \cap X$ is smooth.

■

The cotangent bundle of a projective bundle.

The Euler Sequence: More canonically.

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}^{\oplus n}(-1) \to \mathcal{O} \to 0.$$

If

$$\begin{tikzcd} \mathbb{P}^n_k = \mathbb{P}(V) \arrow[d, "\pi"] \\ \bullet \end{tikzcd}$$

then have:

$V \otimes_k \mathcal{O}_{\mathbb{P}^n_k} \cong \pi^* V \twoheadrightarrow \mathcal{O}(1)$ tautological quotient.

Given: $\pi^* V(-1) \to \mathcal{O}$.

{Note: we are taking here $\mathbb{P}(V) =$ 1-dim'l quotients}.

For projective bundles, we have:

$$\mathbb{P}(\mathcal{E}) \xrightarrow[\pi]{} X.$$

$$\pi^* \mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

Relative Euler sequence: $0 \to \Omega_{\mathbb{P}(\mathcal{E})/X} \to (\pi^* \mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to 0.$

Relative cotangent sequence: $0 \to \pi^* \Omega_{X/k} \to \Omega_{\mathbb{P}(\mathcal{E})/k} \to \Omega_{\mathbb{P}(\mathcal{E})/X} \to 0.$

The canonical bundle

If $X/_{k=\bar{k}}$ smooth

$\Rightarrow \Omega_{X/k}$ locally free of rank $= n = \dim X$

The canonical bundle of X is

$$\omega_X := \Lambda^n \Omega_{X/k} \in \operatorname{Pic} X.$$

Rmk If $X \underset{\varphi}{\cong} X'$ then $\varphi^* \omega_{X'} \underset{\Lambda^n d\varphi}{\cong} \omega_X$.

Example. If $\mathbb{P}(\mathcal{E}) \longrightarrow X$ is a proj. bundle over X smooth $/k = \bar{k}$.

$$\begin{aligned} \omega_{\mathbb{P}(\mathcal{E})} &\cong \det\left(\Omega_{\mathbb{P}(\mathcal{E})/k}\right) \\ &\cong \det\left(\pi^* \Omega_{X/k}\right) \otimes \det\left(\Omega_{\mathbb{P}(\mathcal{E})/X}\right) \\ &\cong \pi^*(\omega_{X/k}) \otimes \det\left(\pi^* \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)\right) \\ &\cong \pi^*\left(\omega_{X/k} \otimes \det \mathcal{E}\right) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-r). \end{aligned}$$

Theorem Let $Y \subset X$ be a closed immersion of smooth varieties $/k=\bar{k}$. Let

$$\pi : \tilde{X} \to X$$

be the blow-up of X at Y. Let $E \overset{i}{\subset} \tilde{X}$ be defined by $\pi^{-1}(I_Y)$.

(A) $\tilde{X}$ is smooth

(B) $E \to Y$ is isomorphic to $\mathbb{P}(I/I^2) \to Y$.

(C) The normal bundle $N_{E/\tilde{X}} \cong i^* \mathcal{O}_{\mathbb{P}(I/I^2)}(-1)$.

Ⓑ

Pf: $\tilde{X} = \mathrm{Proj}(\oplus I^d)$

$$E = \mathrm{Proj}((\oplus I^d)|_Y)$$

$$= \mathrm{Proj}(\oplus I^d/I^{d+1}).$$

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I/I^2 locally free of rank $r = \mathrm{codim}\, Y/X$.

Locally gen'd by a reg. sequence:

$$\Rightarrow I^d/I^{d+1} \cong \mathrm{Sym}^d(I/I^2).$$

Ⓐ So! E is smooth ($\Rightarrow$ regular)

AND the ideal of E is locally principal

$\Rightarrow \tilde{X}$ is regular along E

$\Rightarrow$ smooth!

Ⓒ The ideal of $E \subset \tilde{X}$ is $\mathcal{O}_{\tilde{X}}(1)$

$$\Rightarrow I_E/I_E^2 \cong \mathcal{O}_E(1)$$

$$\Rightarrow N_{E/\tilde{X}} \cong \mathcal{O}_E(-1). \qquad \blacksquare$$

Rmk: (Adjunction Formula)

If $D \subset X$ is a smooth divisor then $\omega_D \cong (\omega_X \otimes \mathcal{O}(D))|_D$.

In particular, in the set-up above:

$$\omega_E \cong (\omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(E))|_E$$

$$\cong \pi^*(\omega_Y \otimes \det I/I^2) \otimes \mathcal{O}_{\mathbb{P}(I/I^2)}(-\underset{\substack{\uparrow \\ \operatorname{codim}(Y \subset X)}}{r}).$$